C^r map :- consider a map $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$

f is C^r map if f has continuous partial derivatives of all order $\leq r$

Thus

f is a C^1-map if it has continuous partial derivative of first order

f is C^2 map if it has continuous partial derivatives upto second order.

f is C^0 map if f is only continuous.

Differential Geometry :- It is something, which locally looks like $\mathbb{R}^n$, in which differentiability concept is defined. Differentiability is the additional property which is not defined in ordinary topological space.

consider a topological space M then every point in M has an nbd, say U which is homeomorphic to some open set of $\mathbb{R}^n$ say $\phi(U)$

Hence it is called locally Eucludean

[illegible] in [illegible] consider $p \in U \subseteq M$ again consi[der]

$\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R}\}$ let u_i is the ith projection map which is linear hence continuous also

Then $u_i \circ \phi(p) = a_i$

now define

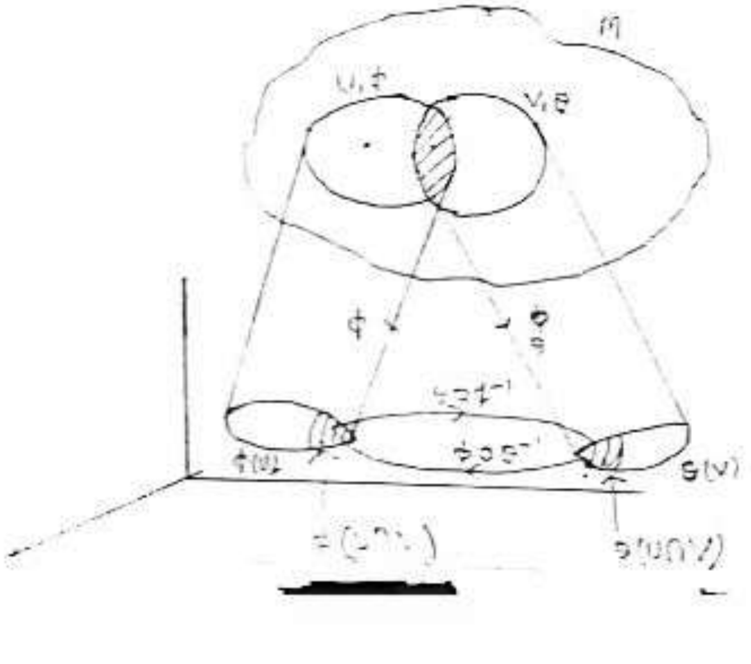
$$x^i(p) = (u_i \circ \phi)(p)$$

$$\text{or } x^i = u_i \circ \phi$$

x^i is called coordinate function

Hence point $p \in U$ can be coordinatize as $(x^1(p), x^2(p), \ldots\ldots, x^n(p))$

Then (U, ϕ) is called Local chart or a Local coordinate system on M



Transition / change of coordinate map :-

let U and V be two open sets in a topological space m

If $p \in U \cap V$ then consider

$(x^1, x^2, \dots, x^n)$ and $(y^1, y^2, \dots, y^n)$ be two different coordimatization of $p \in U \cap V$ by (U, ϕ) and (V, θ) respectively.

But different coordinate of p may disturbe differentiability concept at p so we find a relation or map called transition map between y_i's and x_i's

Define $\theta \circ \phi^{-1} : \phi(U \cap V) \longrightarrow \theta(U \cap V)$

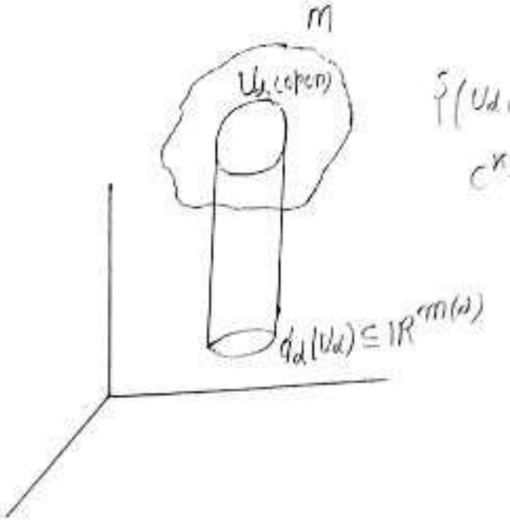
and $\phi \circ \theta^{-1} : \theta(U \cap V) \longrightarrow \phi(U \cap V)$

These are called transition map or change of coordinate maps. Both are C^∞ maps and transform paints of one overlaping region to another

Then pair

$(M, \{(U_\alpha, \phi_\alpha) : \alpha \in I\})$ is called C^∞ manifold

Provided that $\{U_\alpha : \alpha \in I\}$ is an open covering of the space M



$\{(U_\alpha, \phi_\alpha) : \alpha \in J\}$
C^k-atlas on M

<u>Definition</u>:- let M be a hausdroff space. Assume that U_α is an open subset of M and ϕ_α is homeomorphism from open set $U_\alpha \subseteq M$ onto an open subset $\phi_\alpha(U_\alpha)$ of $\mathbb{R}^{m(\alpha)}$ where $m(\alpha)$ is non negative integer and $\alpha \in L$ being indexing set

then collection $\{(U_\alpha, \phi_\alpha) : \alpha \in L\}$ is called C^k-atlas on M if

i) $\bigcup\limits_{\alpha \in L} U_\alpha = M$

ii) if $U_\alpha \cap U_\beta \neq \phi$ then the map

$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U \cap V) \longrightarrow \phi_\alpha(U \cap V)$

and $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U \cap V) \longrightarrow \phi_\beta(U \cap V)$

is a C^k-map $\forall\ \alpha, \beta \in L$ $\quad k \in \mathbb{N} \cup \{\infty\}$

The object $(M, \{(U_\alpha, \phi_\alpha) : \alpha \in L\})$ is called 'C^r-manifold'

The member (U_α, ϕ_α) of a C^k atlas $\{(U_\alpha, \phi_\alpha) : \alpha \in L\}$ are called 'Local chart' or 'coordinate chart' or 'coordinate nbd'

If $p \in U_\alpha$ then take

$$x_\alpha^i(p) = U_i(\phi_\alpha(p))$$ then x_α^i are called

'Local coordinate point' of $p \in U_\alpha$

⇒ The function $x_\alpha^i = U_i \circ \phi_\alpha$ are called 'coordinate function' on U_α

⇒ The pair (M, F) is generally used to denote C^k manifold.

⇒ An atlas on M is called smooth atlas if it is C^k atlas $\forall\, k \in \mathbb{N}$

⇒ A Hausdorff topological space together with a smooth atlas is called Smooth manifold or Differentiable manifold

⇒ If we assume M to be connected then all $m(\alpha)$ will be neccessarly same by inverse

mapping theorem. in this case common inte[ger]
say 'm' is called dimension of manifold M

$\Rightarrow$ sphere is locally homeomorphic to $\mathbb{R}^2$

$\Rightarrow$ circle are curve is locally homeomorphic to $\mathbb{R}$

Hence surfaces is two dimensional manifold

example-(1) let $M = \mathbb{R}^m$ then map

$$id : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$

given by $id(x) = x \quad \forall x \in \mathbb{R}^m$

is a homeomorphism, so that $(\mathbb{R}^m, id)$ with identity maps is a gobal chart on $\mathbb{R}^m$

consider $\{(\mathbb{R}^m, id)\}$ is a C^k-atlas on $\mathbb{R}^m$ $\forall k \in \mathbb{N}$

so $(\mathbb{R}^m, \{(\mathbb{R}^m, id)\})$ is a smooth manifold.

example-2 :- let $M \subseteq \mathbb{R}^m$ be an open subset

let $\phi = id/M$

then $(\mathbb{R}^m, \phi)$ is a gobal chart on M and so $\{\cancel{\mathbb{R}^m}(M, \phi)\}$ is a C^k-atlas ~~on M~~ $\forall k \in \mathbb{N}$ on M and so the pair $(M, \{(M, \phi)\})$ is a smooth manifold

$$S^n = \left\{ (x_1, x_2, \ldots\ x_{n+1}) \in \mathbb{R}^{n+1} ;\ \sum_{i=1}^{n+1} x_i^2 = 1 \right\}$$

(is a sphere in higher dimension)

$$S^1 = \left\{ (x_1, x_2) \in \mathbb{R}^2 ;\ x_1^2 + x_2^2 = 1 \right\}$$

$$S^2 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 :\ x_1^2 + x_2^2 + x_3^2 = 1 \right\}$$

<u>Differentiable structure</u>: If the collection satisfies $\left\{ (U_\alpha, \phi_\alpha) : \alpha \in \Lambda \right\}$

i) $\bigcup_{\alpha \in \Lambda} U_\alpha = M$

ii) If $U_\alpha \cap U_\beta \neq \phi$ then the map

$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta (U_\alpha \cap U_\beta) \longrightarrow \phi_\alpha (U_\alpha \cap U_\beta)$

and $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha (U_\alpha \cap U_\beta) \longrightarrow \phi_\beta (U_\alpha \cap U_\beta)$

is a C^k-maps $\forall\ \alpha, \beta \in \Lambda$ $\quad k \in \mathbb{N} \cup \{\infty\}$

iii) above collection is maximal w.r.t. (i) then manifold is called a differential structure on M

ex:- $\quad M = \mathbb{R}^n$

$k \neq 0$

$\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\bar{x} \longrightarrow k\bar{x}$

$(\phi \circ id^{-1})(x) = \phi(x) = kx$

$(id \circ \phi^{-1})(x) = id\, \phi^{-1}(x) = x/k$

$\Rightarrow$ maximality are not satisfies

- A space may admit more than one differentiable structure.

ex:- let $M = \mathbb{R}$ than $\{(\mathbb{R}, id)\}$ provides a diff structure on $M = \mathbb{R}$ which is called standared differentiable structure.

let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be given as

$$\phi(x) = x^3 \quad \forall\, x \in \mathbb{R}$$

then $(\mathbb{R}, \phi)$ is global chart and so $\{(\mathbb{R}, \phi)\}$ prov a differential structure on $\mathbb{R}$

for check

$$\phi^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longrightarrow x^{1/3}$$

$(\phi \circ id^{-1})(x) = \phi(id^{-1}(x)) = \phi(x) = x^3$ which is C^{∞}

$(id \circ \phi^{-1})(x) = id(\phi^{-1}(x)) = id(x^{1/3}) = x^{1/3}$

which is not C^1- maps but it is C^0- maps.

thus $(\mathbb{R}, \phi)$ is not compatible with $(\mathbb{R}, id)$

so both differentiable structure on $\mathbb{R}$ is differen ex:- let V be a n-dimensional vector space over $\mathbb{R}$

let $\{e_1, e_2, \ldots\ldots e_n\}$ be an ordered basis of $\mathbb{R}^n$

then for each $v \in V$ $\exists$ unique real numbers

$\alpha_1, \alpha_2, \ldots\ldots \alpha_n$ s.t

$$v = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \ldots\ldots + \alpha_n e_n = \sum_{i=1}^{n} \alpha_i e_i$$

$\alpha_1, \alpha_2, \ldots\ldots \alpha_n$ are called coordinate of vector v relative to ordered basis

let us define a map

$$\phi : V \longrightarrow \mathbb{R}^n$$

$$\phi(v) = (\alpha_1, \alpha_2, \ldots\ldots \alpha_n)$$

where $v = \sum_{i=1}^{n} \alpha_i e_i$

then ϕ is an isomorphism i.e

$$V \cong \mathbb{R}^n \qquad (\because \text{ as for } V^n(\mathbb{R}) \quad V \cong \mathbb{R}^n)$$

check : ϕ is homoeomorphism (is continuous + bijective)

and show that $\{(V, \phi)\}$ is a C^k-atlas $\forall\, k = 1, 2 \ldots$

So V with above atlas is a C^k-manifold $\forall\, k = 1, 2, \ldots\ldots$ of dimension ~~on~~ ~~n~~ n ex: consider

$M(n, \mathbb{R}) = \{X : (x_{ij}) : x_{ij} \in \mathbb{R} \quad 1 \le i, j \le n\}$

i.e the set of all real matrices which is vector space over $\mathbb{R}$ of dimension n^2

Define a map

$$\theta : M(n, \mathbb{R}) \longrightarrow \mathbb{R}^{n^2} \text{ as}$$

$$(x_{ij})_{n \times n} \longrightarrow (x_{11}, x_{12}, \dots x_{1n}, \dots \quad x_{n1}, x_{n2}, \dots x_{nn}) \in \mathbb{R}^{n^2}$$

then θ is an isomorphism and so

$$M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$$

then $M(n, \mathbb{R})$ is differentiable manifold of dimension n^2 (as the vector space is finite dim at real field.

ex:- $GL(n, \mathbb{R}) = \{A \in M(n, \mathbb{R}) \mid A^{-1} \text{ exist}\}$

$$= \{A \in M(n, \mathbb{R}) \mid |A| \neq 0\}$$

consider the determinant map

$$\det : M(n, \mathbb{R}) \longrightarrow \mathbb{R} \quad \text{as}$$

$$\det A = |A|$$

then $GL(n, \mathbb{R})$

$\mathbb{R} \setminus \{0\}$ is an op

n map is polynomi

continuous

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} -$$

Hence $GL(n, \mathbb{R})$

$M(n, \mathbb{R}) \cong \mathbb{R}^n$

So $GL(n, \mathbb{R})$ is c

(as any open

ex:- let us

$S^n = \{($

S^1 - is c

$S^2 = \{(x_1$

then $GL(n, \mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$

$\mathbb{R} \setminus \{0\}$ is an open subset of $\mathbb{R}$ and as det map is polynomial in it's entries so it is continuous

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{\det} \underbrace{a_{11}a_{22} - a_{21}a_{12}}_{\text{polynomial}}$$

Hence $GL(n, \mathbb{R}) = \det^{-1}\{\mathbb{R} \setminus \{0\}\}$ must be open in $M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$

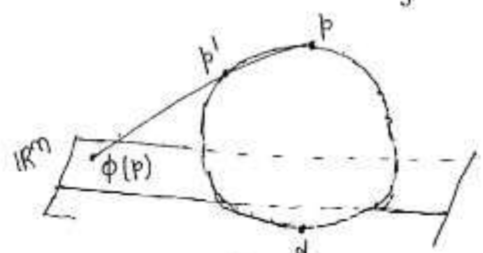
So $GL(n, \mathbb{R})$ is a differential manifold of dimension n^2 (as any open subset of $\mathbb{R}^n$ is differentiable manifold)

ex:— let us consider n-sphere

$$S^n = \left\{ (x_1, x_2, \ldots\ldots x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1 \right\}$$

S^1 – is called unit circle

$$S^2 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \right\}$$



$U = S^n \setminus \{p\}$

$V = S^n - \{q\}$

$U \cup V = S^n$

$\phi : U \longrightarrow \mathbb{R}^n$

$\psi : V \longrightarrow \mathbb{R}^n$

let $p = e_{n+1} = (0,0,0,\ldots\ldots,1)$ i.e north pole this point
of sphere S^n)
if $t(x_{n+1}-1)+1$
and $q = -e_{n+1} = (0,0,\ldots\ldots -1)$ (i.e south pole of $t = $
sphere S^n)

Take $U = S^n \setminus \{p\}$ and $V = S^n \setminus \{q\}$ let ϕ be so $\phi(x_1, x_2, \ldots$
stereographic projection from p onto the plane $\mathbb{R}^n$ similarly we
$\mathbb{R}^n = \left\{ \bar{x} = (x_1, x_2, \ldots x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = 0 \right\}$

let $p' \in U$, then $\phi(p')$ is the point of intersection $\psi(x_1, x_2, \ldots$
of the line joining p and p' with $\mathbb{R}^n = \{\bar{x} \in \mathbb{R}^{n+1} : x_{n+1} =$ For invers
we define ψ is stereographic projection from q to $\mathbb{R}^n$ $\phi^{-1} : \mathbb{R}^n =$
expression for $\phi : U \longrightarrow \mathbb{R}^n = \{\bar{x} \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ let $\bar{x} =$
$\bar{x}$ wit
let $p' = (x_1, x_2, \ldots\ldots x_{n+1}) \in U$
$\bar{\alpha}(t$
the equation of line joining p and p' is given as

$$\bar{\alpha}(t) = (0, 0, \ldots\ldots 0, 1) + t(x_1, x_2, \ldots\ldots x_{n+1} - 1)$$
$$= (tx_1, tx_2, \ldots\ldots tx_n, t(x_{n+1} - 1) + 1)$$

Scanned with CamScanner

north pole this point lies on the plane $x_{n+1} = 0$

iff

$$t(x_{n+1} - 1) + 1 = 0$$

th pole of

$$t = \frac{1}{1 - x_{n+1}}$$

So

be

$$\phi(x_1, x_2, \ldots x_n, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, x_2, \ldots x_n, 0)$$

plane $\mathbb{R}^n$ similarly we can find

$= 0\}$

$$\psi : V \longrightarrow \mathbb{R}^n \text{ as}$$

$$\psi(x_1, x_2, \ldots x_n, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, x_2, \ldots, x_n, 0)$$

tersection

$\in \mathbb{R}^{n+1} : x_{n+1} =$

For inverse map ϕ^{-1}

$$\phi^{-1} : \mathbb{R}^n = \{\bar{x} \in \mathbb{R}^{n+1} : x_{n+1} = 0\} \longrightarrow U$$

m ϕ to $\mathbb{R}^n$

$x_{n+1} = 0\}$

let $\bar{x} = (x_1, x_2, \ldots, x_n, 0) \in \mathbb{R}^n$ then line joining $\bar{x}$ with e_{n+1} is given by

$$\bar{\alpha}(t) = t\bar{x} + (1 - t)e_{n+1}$$

given as

$$= t(x_1, x_2, \ldots x_n, 0) + (1-t)(0, 0, \ldots 0, 1)$$

$x_{n+1} - 1)$

$$= (tx_1, tx_2, \ldots tx_n, 1 - t)$$

$) + 1)$

$$= (t\bar{x}', 1 - t)$$

where $\bar{x}' = (x_1, x_2, \ldots x_n)$

The point $\bar{x}(t)$ lies on S^n iff

$$\|\bar{x}(t)\|^2 = 1$$

$$t^2\|\bar{x}'\|^2 + (1-t)^2 = 1$$

i.e $$t^2\|\bar{x}'\|^2 + 1 + t^2 - 2t = 1$$

$$(1+\|\bar{x}'\|^2)t^2 = 2t$$

$$t = \frac{2}{1+\|\bar{x}'\|^2}. \quad t \neq 0$$

So $\phi^{-1}: \mathbb{R}^n \longrightarrow U$ is given as

$$\phi^{-1}(x_1, x_2, \ldots x_n, 0) = \bar{x}(t)$$

$$= (t\bar{x}', 1-t)$$

$$\phi^{-1}(x_1, x_2, \ldots x_n, 0) = \frac{1}{1+\|\bar{x}'\|^2}(2x_1, 2x_2, \ldots 2x_n, \|\bar{x}'\|^2 - 1)$$

similarly $\psi^{-1}: \mathbb{R}^n \longrightarrow V$ is given by

$$\psi^{-1}(x_1, x_2, \ldots x_n, 0) = \frac{1}{1+\|\bar{x}'\|^2}(2x_1, 2x_2, \ldots 2x_n, 1 - \|\bar{x}'\|^2)$$

now $U \cap V = S^n \setminus \{p, q\}$

$\phi(U \cap V) = \psi(U \cap V) = \mathbb{R}^n \setminus \{0\}$

Scanned with CamScanner

thus

$$\phi \circ \psi^{-1}(x) = \|x\|^{-2}(x_1, x_2, \ldots x_n, 0)$$
$$\psi \circ \phi^{-1}(x) = \|x\|^{-2}(x_1, x_2, \ldots x_n, 0)$$

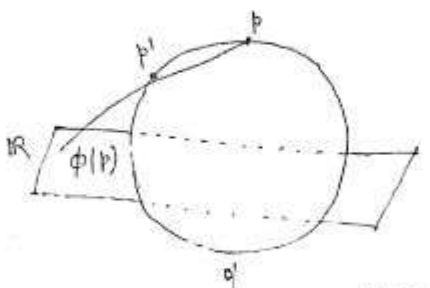
which are C^k maps $\forall k$

thus $\{(U, \phi), (V, \psi)\}$ is a C^k-atlas on $\mathbb{R}^n$ for all k

nd so $\left(S^n, \{(U, \phi), (V, \psi)\}\right)$ is a smooth manifold.

Ex: show that S^1 is differentiable manifold

Soln



$U = S^1 \setminus \{p\}$

$V = S^1 - \{q\}$

$U \cup V = S^n$

$\phi : U \longrightarrow \mathbb{R}$

$\psi : V \longrightarrow \mathbb{R}$

let $p = e_2 = (0, 1)$ [ie north pole of circle S^1)

and $q = -e_2 = (0, -1)$ [$\because$ south pole of ~~cir~~ circle S^1)

Take $U = S^1 - \{p\}$ and $V = S^1 - \{q\}$

Let ϕ be stereographic projection from p onto
$\mathbb{R} = \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_2 = 0 \}$

Let $p' \in U$ then $\phi(p')$ is the point of intersection of line joining p and p' with $\mathbb{R} = \{ x \in \mathbb{R}^2 : x_2 = 0 \}$
we define ϕ is stereographic projection from
$\mathbb{R}$

expression for $\phi : U \longrightarrow \mathbb{R} = \{ \bar{x} \in \mathbb{R}^2 : x_2 = 0 \}$

let $p' = (x_1, x_2) \in U$

The equation of line joining p and p' is given as

$$\bar{x}(t) = (0, 1) + t(x_1, x_2 - 1)$$

$$= \left(t x_1 , t(x_2 - 1) + 1 \right)$$

This point lies on plane $\mathbb{R}$ $x_2 = 0$ if

$$t(x_2 - 1) + 1 = 0$$

$$t = \frac{1}{1 - x_2}$$

and so

$$\phi(x_1, x_2) = \frac{1}{1 - x_2} (x_1)$$

$$\phi(x_1, x_2) = \frac{1}{1 - x_2} (x_1, 0)$$

Scanned with CamScanner

similarly we can find

$$\psi : V \longrightarrow \mathbb{R}^{0} \text{ as}$$

$$\psi(x_1, x_2) = \frac{1}{1+x_2}(x_1, 0)$$

For inverse map ϕ^{-1}.

$$\phi^{-1} : \mathbb{R} - \{ \bar{x} \in \mathbb{R}^2 : x_1 \; 0 \} \longrightarrow U$$

if $\bar{x}$ $(x_1, 0) \in \mathbb{R}$ then line joining $\bar{x}$ with e_2 is given by

$$\bar{z}(t) = t\bar{x} + (1-t)e_2$$

$$= t(x_1, 0) + (1-t)(0,1)$$

$$= (tx_1, 0) + (0, 1-t)$$

$$= (tx_1, 1-t)$$

The point $\bar{z}(t)$ lies on S^1 if

$$\|\bar{z}(t)\|^2 = 1$$

$$t^2\|x_1\|^2 + (1-t)^2 = 1 \quad \Rightarrow t^2x_1^2 + \cancel{1} + t^2 - 2t = \cancel{1}$$

$$t^2\|x_1\|^2 \qquad t^2(1+x_1^2) = 2t$$

$$t = \frac{2}{1+x_1^2} \qquad t \neq 0$$

so $\phi^{-1}\; \mathbb{R} \longrightarrow U$ is given as

$$(x_1, 0) = \bar{z}(t)$$

$$= (tx_1, 1-t)$$

$\phi^{-1}(x_1, 0)$ … = …

Similarly

$\psi^{-1} : \mathbb{R} \longrightarrow V$ is given by

$$\psi^{-1}(x_1, 0) = \frac{1}{1+x_1^2}(2x_1, 1-x_1^2)$$

now $U \cap V = S^1 - \{p, q\}$

now $\phi(U\cap V) = \psi(U \cap V) = \mathbb{R} \setminus \{0\}$

$$\phi \circ \psi^{-1}(x) = \phi\left[\frac{1}{1+x_1^2}(2x_1, 1-x_1^2)\right]$$

$$= x_1^{-2}(x_1, 0)$$

$$\psi \circ \phi^{-1}(x) = x_1^{-2}(x_1, 0)$$

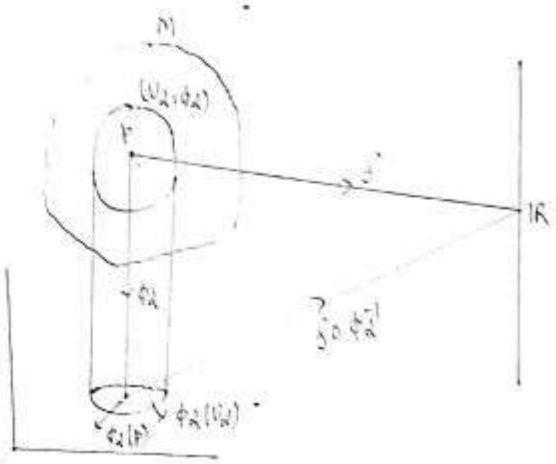
which are C^K-maps $\forall K$

Thus $\{(U,\phi), (V,\psi)\}$ is a C^K-atlas on $\mathbb{R}$ $\forall K$

and so $\left(S^1, \{(U,\phi), (V,\psi)\}\right)$ is a smooth manifold.

smooth function.

Definition :- let M be smooth manifold $f: M \longrightarrow \mathbb{R}$ is a map then f is said to be smooth at $p \in M$ if whenever p lies in a local chart (U_α, ϕ_α), then map $f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is smooth where $n = \dim(M)$



we observe that above definition is independent of choice of local chart in which p lies for if (U_β, ϕ_β) is another local chart in which p lies then we can write.

$$f \circ \phi_\alpha^{-1} = (f \circ \phi_\beta^{-1}) \circ (\phi_\beta \circ \phi_\alpha^{-1})$$
$$f \circ \phi_\beta^{-1} = (f \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \phi_\beta^{-1})$$

So $f \circ \phi$ $f \circ \phi_\alpha^{-1}$ is smooth iff $f \circ \phi_\beta^{-1}$ is smooth

A function $f : \overset{\text{open}}{U \subseteq M} \longrightarrow \mathbb{R}$ is said to be smooth smooth on

if it is smooth at each point p of U.

<u>example - (1)</u> :- let $f : M \longrightarrow \mathbb{R}$ be defined as

$$f(p) = c \quad \forall\ p \in M$$

where c is some fixed real number.

then f is smooth.

<u>example-2</u> let M be an m-dimensional smooth manifold and (U_α, ϕ_α) be local chart of M

let $x^i = u_i \circ \phi_\alpha$ (local coordinate function)

then x^i $(i = 1, 2, \ldots, m)$ are smooth of U_α

$$x^i \circ \phi_\alpha^{-1} = (u_i \circ \phi_\alpha) \circ \phi_\alpha^{-1}$$
$$= u_i \quad (\text{smooth})$$

Hence local smooth function exist.

<u>construction of globally smooth function on M</u> :-

let F be any smooth function on $\phi_\alpha(U_\alpha)$

then $f = F \circ \phi_\alpha$ smooth on U_α

$$f \circ \phi_\alpha^{-1} = (F \circ \phi_\alpha) \circ \phi_\alpha^{-1} = F$$

let u be an

ϕ_α

$f = Fc$

Define $g : M$

$g :$

and

then g is sr

let $C^\infty(M)$ den

on M

for $f, g \in U$

$(f+g)$

at (cf)

then $C^\infty(r$

<u>Definition</u> :-

function

If there

(V_α, ψ)

and

$\psi \circ f \circ \phi$

let u be an open subset of M and F be smooth on $\phi_2(U_2)$ then

$f = F \circ \phi$ is smooth on u

Define $g : M \longrightarrow \mathbb{R}$ as

$g = f$ on u

and $g = 0$ on M

then g is smooth on M

Note

let $C^{\infty}(M)$ denote the set of all smooth function on M

for $f, g \in U$ and $c \in \mathbb{R}$ define

$$(f+g)(p) = f(p) + g(p)$$

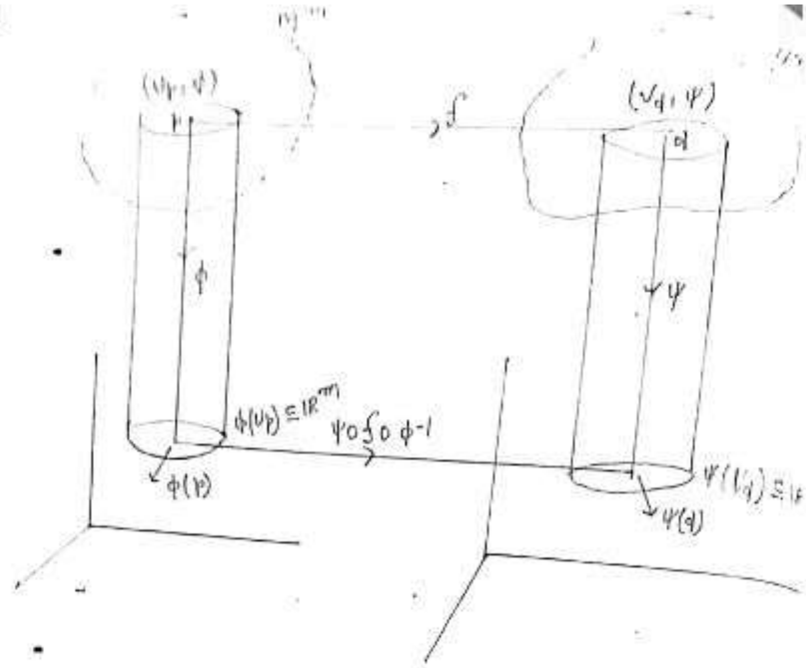
at $(cf)(p) = cf(p)$

Then $C^{\infty}(M)$ is a vector space over $\mathbb{R}$

Definition:- let M and N be smooth manifold of dimension m and n respectively then

function $f : U \subseteq M \longrightarrow N$ is said to be smooth at $p \in U$ If there exist local chart (U_p, ϕ) around p and (V_q, ψ) around $q = f(p)$ such that $f(U_p) \subseteq V_q$ and the map

$\psi \circ f \circ \phi^{-1} : \phi(U_p) \subseteq \mathbb{R}^m \longrightarrow \psi(V_q) \subseteq \mathbb{R}^n$ is smooth.



<u>example</u>- let m be the manifold IR with u^3- atlas

Define $f : m \longrightarrow IR$ as

$$f(t) = t \quad \forall\, t \in m$$

$\phi : IR \longrightarrow IR$ given as $\phi(t) = t^3$

If f is smooth

$\phi^{-1} : IR \longrightarrow IR$ is given by

$$\phi^{-1}(t) = t^{1/3}$$

now $(f \circ \phi^{-1})(t) = f[\phi^{-1}(t)]$

$$= f[t^{1/3}]$$

$$= t^{1/3}$$

As $f \circ \phi^{-1}$ is not smooth so f is not smooth.

ex1 define $f: M \longrightarrow \mathbb{R}$ as

$$f(t) = t^3 \quad \forall t \in M$$

and $\phi: \mathbb{R} \longrightarrow \mathbb{R}$ given as

$$\phi(t) = t^3$$

If f is smooth

$$\phi^{-1}: \mathbb{R} \longrightarrow \mathbb{R} \text{ is given by}$$

$$\phi^{-1}(t) = t^{1/3}$$

now $(f \circ \phi^{-1})(t) = f[\phi^{-1}(t)]$

$$= f(t^{1/3})$$

$$= t$$

As $f \circ \phi^{-1}$ is smooth so f is smooth.

<u>example</u>:- let M be the manifold $\mathbb{R}$ with usual smooth structure and N be the manifold $\mathbb{R}$ with u^3 structure

Define a map $f: M \longrightarrow N$ as

$$f(t) = t^{1/3}$$

$$\phi: \mathbb{R} \longrightarrow \mathbb{R} \text{ as } \phi(t) = t^3$$

$$\phi^{-1}: \mathbb{R} \longrightarrow \mathbb{R} \text{ as } \phi^{-1}(t) = t^{1/3}$$

Is f smooth

consider map $(\phi \circ f \circ id^{-1})(t) = \phi \circ f[id^{-1}(t)]$

$$= (\phi \circ f)(t)$$

$= \phi[\psi(t)]$

$= \phi(t^{1/3})$

$= t$

which shows that $\phi \circ id^{-1}$ is smooth & f^{-1} is [illegible] to smooth

ex:- let $\phi : M \longrightarrow N$ and $\psi : N \longrightarrow S$ be smooth maps

Then $\psi \circ \phi : M \longrightarrow S$ is also smooth.

<u>Diffeomorphism</u> :- A map $\phi : M \longrightarrow N$ is said to be diffeomorphism if is smooth bijective and $\phi^{-1} : N \longrightarrow M$ is also smooth then two manifold M and N are said to be diffeomorphic.

<u>ex: (i)</u> : let $\mathbb{R}$ be smooth manifold with usual structure

define a map $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ as

$$\phi(t) = t^3 \quad t \in \mathbb{R}$$

Then ϕ is a homeomorphism but it is not diffeomorphism.

example-2 :- let M be a manifold $\mathbb{R}$ with standerd differential structure and N is u^3 also structure

Define a map $f : M \longrightarrow N$ by

$$f(t) = t^{1/3} \quad \forall\, t \in M$$

If f is diffeomorphism

$$\psi \circ f \circ \phi^{-1}(t) = (\psi \circ f)(\phi^{-1}(t))$$

$$= (\psi \circ f)(t)$$

$$= \psi(f(t))$$

$$= \psi(t^{1/3})$$

$$= t$$

since $\psi \circ f \circ \phi^{-1}$ is smooth so f is also smooth

$\psi^{-1} : \mathbb{R} \longrightarrow N$ is given as

$$\psi^{-1}(t) = t^{1/3} \quad \forall\, t \in \mathbb{R}$$

now $$(\phi \circ f^{-1} \circ \psi^{-1})(t) = (\phi \circ f^{-1})(\psi^{-1}(t))$$

$$= (\phi \circ f^{-1})(t^{1/3})$$

$$= \phi\left[f^{-1}(t^{1/3})\right]$$

$$= \phi(t)$$

$$= t$$

since $\phi \circ f^{-1} \circ \psi^{-1}$ is smooth so f^{-1} is also smooth

Hence f is diffeomorphism of M onto N

we observe that